OBSERVATIONS IN RELIABILITY ANALYSIS OF A 1-D CONSOLIDATION PROBLEM

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ABSTRACT

This paper presents the results of a reliability analysis for a one-dimensional consolidation problem. The peculiarity of this problem is that there are two local solutions for the design points, depending on the initial point of the search algorithm in the first-order reliability method, probably due to the non-differentiability of the performance function. One of the local design points is the global solution of the first-order reliability method, while the other is a fake one. Unfortunately, it turns out to be relatively easy to find the fake solution in certain scenarios. This phenomenon of multiple design points is studied and documented in detail in this paper. The analysis results by using other reliability methods are also presented for comparison. Finally, recommendations are given for the suitable reliability methods for this particular example. The conclusion of this paper suggests that cautions should be used when implementing any design-point based reliability method to problems with non-differentiable performance functions.

Key words: Reliability, consolidation, first-order reliability method, performance function.

1. INTRODUCTION

It is well known that many uncertainties are present in geotechnical engineering. As a consequence, a safety factor greater than unity does not guarantee safety. In the past, safety margins (*i.e.*, safety factor > 1) are the primary way of accommodating and quantifying these uncertainties. More recently, reliability, namely one minus failure probability, has been taken as a more rigorous quantification of geotechnical uncertainties (Christian, *et al*., 1994; Low, *et al*., 1998; Zhang, *et al*., 2001; Phoon, *et al*., 2003; Fenton and Griffiths, 2003; Chalermyanont and Benson, 2004).

Many reliability methods are originally developed by researchers in structural engineering, *e.g.* first- and second-order reliability methods (Ang and Tang, 1984; Der Kiureghian, *et al*., 1987; Melchers, 1999), first-order second moment methods (Ang and Tang, 1984), *etc*. Their applicability to geotechnical problems should be examined because geotechnical models may have different characteristics from structural ones. For instance, spatial variability in ground is usually quite pronounced, but this may not be the case in structural components; constitutive and governing equations of geotechnical materials can be highly nonlinear or even discontinuous, while structural components may behave linearly in small stress conditions. Direct use of reliability methods originated from structural engineering for geotechnical problems without verification can be dangerous and giving misleading conclusions.

This paper addresses the issue of non-differentiability in performance functions that is often encountered in geotechnical problems. In particular, a simple one-dimensional consolidation problem will be taken to demonstrate the effect of this non-differentiability in reliability analyses. Such non-differentiability clearly exists in a one-dimensional consolidation problem because of the switching between normally consolidation and over-consolidation regimes. The focus will be placed at the phenomenon of multiple design points, which are needed in the first-order reliability method (FORM) and second-order reliability method (SORM). The conclusion is that FORM and SORM can give misleading analysis results due to the issue of multiple design points although they are among the most popular reliability methods. Asides from FORM and SORM, other popular reliability methods including first-order second-moment method, Monte Carlo simulation (Ang and Tang, 1984) and subset simulation (Au and Beck, 2001) will be also examined for their consistency over problems with non-differentiable performance functions.

Because FORM is very popular, attempts are made to propose FORM-based reliability methods that are able to resolve this issue of multiple design points, but one should be cautious that the proposed method may only work well for consolidation problems. For other geotechnical problems with non-differentiable performance function, the proposed method may not work at all.

The purpose of this paper is not to make general conclusions regarding problems with non-differentiable performance functions but to demonstrate a "warning" example to alert geotechnical researchers and practicing engineers to use cautions in choosing reliability methods. The structure of this paper is as follows. First the consolidation example will be presented, and adopted reliability methods described. Results of reliability analyses will be given, and a preliminary comparison of the results be made. The issue of multiple design points pertaining to FORM and SORM will be examined in detail, and its link with the non-differentiability be discussed. Sensitivity analysis will be taken to further understand the severity of the issue under various scenarios. Finally, attempts are made to suggest FORM-based methods that are able to resolve the issue.

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2. DESCRIPTION OF THE CONSOLIDATION PROBLEM

Consider a saturated clay layer of thickness H between two sand layers, as shown in Fig. 1. The surcharge pressure at the ground surface is q. The clay is lightly overconsolidated with an uncertain OCR. The compression C_c and recompression C_r indices of the clay are also uncertain. It is assumed that the recompression index is an uncertain fraction α of the compression index:

$$C_r = \alpha\, C_c \tag{1}$$

The failure is defined as the consolidation settlement exceeds a prescribed allowable settlement S_{allow}. In other words, the performance function G can be written as ($G < 0$ defines failure):

$$G(X) = \begin{cases} S_{\text{allow}} - \dfrac{H}{1+e^{\text{clay}}}\left[C_r \log\left(\dfrac{\sigma'_p}{\sigma'_0}\right) + C_c \log\left(\dfrac{\sigma'}{\sigma'_p}\right)\right] & \text{if } \sigma' \ge \sigma'_p \\ S_{\text{allow}} - \dfrac{H \cdot C_r \log_{10}\left(\dfrac{\sigma'}{\sigma'_0}\right)}{1+e^{\text{clay}}} & \text{if } \sigma' < \sigma'_p \end{cases} \tag{2}$$

where X contains all uncertain variables; e^{clay} is the initial void ratio of the clay; σ'_0 and σ' are the effective consolidation stresses before and after the surcharge is applied; σ'_p is the preconsolidation stress;

$$\sigma' = \sigma'_0 + q \;; \qquad \sigma'_p = \text{OCR} \cdot \sigma'_0 \tag{3}$$

and

$$\sigma'_0 = 0.5\gamma^{\text{sand}} + 1(\gamma^{\text{sand}}_{\text{sat}} - \gamma_w) + \frac{H}{2}(\gamma^{\text{clay}}_{\text{sat}} - \gamma_w) \tag{4}$$

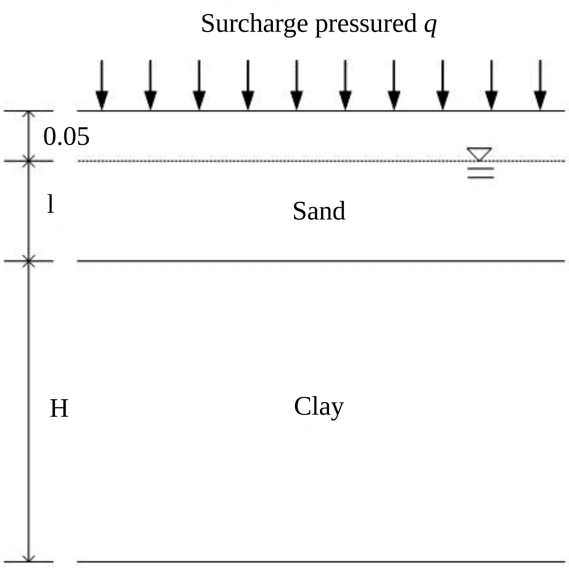

Fig. 1 The consolidation problem

where γ and γ_{sat} are the moist and saturated units weights of the soils; γ_w = 9.8 kN/m^2 is the unit weight of water. The moist and saturated soil unit weights are not independent, because they are related to the specific gravity of the soil solids G_s and the void ratio e:

$$\gamma^{\text{sand}}_{\text{sat}} = \gamma_w \frac{G_s^{\text{sand}} + e^{\text{sand}}}{1+e^{\text{sand}}} \;; \quad \gamma^{\text{sand}} = \gamma_w \frac{G_s^{\text{sand}} + 0.2e^{\text{sand}}}{1+e^{\text{sand}}} \;;$$
$$\gamma^{\text{clay}}_{\text{sat}} = \gamma_w \frac{G_s^{\text{clay}} + e^{\text{clay}}}{1+e^{\text{clay}}} \tag{5}$$

where a degree of saturation = 20% is assumed for "moist". There are nine independent random variables in this problem, *i.e.*, X contains q, H, e^{clay}, e^{sand}, G_s^{clay}, G_s^{sand}, OCR, C_c, and α. The performance function $G(X)$ is not differentiable at the boundary $\sigma' = \sigma'_p$.

The values and distributions of the basic input variables are summarized in Table 1. The allowable settlement S_{allow} is taken to be 0.05 m for the time being.

3. ADOPTED RELIABILITY METHODS

The reliability methods employed in this study include the first-order second moment (FOSM), first-order reliability method (FORM) (Ang and Tang, 1984), second-order reliability method (SORM) (Der Kiureghian and Stefano, 1991), direct Monte Carlo simulation (MCS) and subset simulation (Subsim) (Au and Beck, 2001). What follows briefly reviews these methods.

3.1 FOSM

Under the assumption that $G(X)$ is normally distributed, the failure probability has the following analytical expression:

$$P\big(G(X) < 0\big) = P(\mu_G + \sigma_G Z < 0) = \Phi\left(\frac{-\mu_G}{\sigma_G}\right) = \Phi(-\beta) \tag{6}$$

Table 1 The values and distributions of the input variables

Variable	Distribution	Statistics
S_{allow}	Deterministic	0.05 m
q	Lognormal	mean = 20 kN/m^2; cov* = 20%
H	Gaussian	mean = 4 m; cov = 10%
e^{clay}	Lognormal	mean = 1.2; cov = 15%
e^{sand}	Lognormal	mean = 0.8; cov = 15%
G_s^{clay}	Uniform	[2.5, 2.7]
G_s^{sand}	Uniform	[2.5, 2.7]
OCR	Uniform	[1.5, 2.5]
C_c	Lognormal	mean = 0.4; cov = 25%
α	Uniform	[0.1, 0.2]

* "cov" stands for coefficient of variation.

where μ_G and σ_G are the mean value and standard deviation of $G(X)$; Z is standard Gaussian; Φ is the cumulative density function of the standard Gaussian distribution; β is called the reliability index. The values of μ_G and σ_G can be estimated with Taylor series expansion of the $G(X)$ function around the mean value of X by further assuming $G(X)$ is linear in X.

3.2 FORM/SORM

The basis of FORM is the observation that the volume under the standard Gaussian distribution over the half space defined by a hyperplane in the standard Gaussian space equals to $\Phi(-\beta)$, where the reliability index β is exactly the minimum distance of the hyperplane to the origin. Let $g(Z)$ be the performance function transformed to the standard Gaussian space (*i.e.*, Z is jointly standard Gaussian). In FORM, $g(Z)$ is assumed to be linear in Z, *i.e.*, $g(Z) = 0$ defines a hyperplane, the failure probability is clearly

$$P\left(G(X)<0\right) = P\left(g(Z)<0\right) = \Phi(-\beta) \tag{7}$$

where β is the minimum distance of the hyperplane $g(Z) = 0$ to the origin. Mathematically, if a point z^* on the $g(Z) = 0$ surface that is closest to the origin can be found, the distance between z^* and the origin is exactly the reliability index β. Such a point z^* is named the design point.

As a consequence, the problem of determining failure probability becomes equivalent to finding the design point. There are numerous algorithms of finding the design point; most of them involve a constrained optimization in the standard Gaussian space. In the following analysis, FORM is implemented with the gradient projection (GP) algorithm (Liu and Der Kiureghian, 1991) for the search of the design point. This algorithm involves successive updates of the solution point in the tangential and normal directions of the limit state line $g(Z) = 0$ while keeping the update point on the limit state line.

On the other hand, SORM assumes $g(Z) = 0$ is a quadratic surface in Z. In principle, SORM uses the result the design point obtained from FORM and uses a quadratic function of Z to best match the $g(Z) = 0$ surface. The failure probability can then be found by analytical solution. In the following analysis, SORM is implemented with the algorithm developed by Der Kiureghian and Stefano (1991).

3.3 Mont Carlo Simulation

Mont Carlo simulation is probably the most robust reliability method among all. It is based on the fact that the failure probability is the expected value of the indicator function of failure:

$$P\left(g(Z)<0\right) = E\left(I\left[g(Z)<0\right]\right) \tag{8}$$

where $I[\cdot]$ is the indicator function: if $g(z) < 0$, $I[g(z) < 0]$ is unity, otherwise it is zero. The Law of Large Number states that

$$P\left(g(Z)<0\right) \approx \frac{1}{N}\sum_{i=1}^{N} I\left[g(Z^i)<0\right] \tag{9}$$

where Z^i is the i-th Monte Carlo sample of Z. Although MCS is robust, its failure probability estimate is inaccurate if the actual failure probability is small. The coefficient of variation (cov) for MCS is $[(1-P_F)/(nP_F)]^{0.5}$, where n is the total number of MCS samples and P_F is the actual failure probability. To increase the degree of accuracy for small P_F, one has to take many samples, *i.e.*, n must be large, in MCS, hence MCS can be very computational demanding.

3.4 Subsim

Subset simulation is also a robust reliability method, but it is less computational demanding than MCS, even in the case of small failure probability. The simple but pivotal idea behind Subsim is that a small failure probability can be expressed as a product of larger conditional failure probabilities for some intermediate failure events, suggesting the possibility of converting a problem involving rare-event simulation into a sequence of problems involving more frequent events. This idea can be expressed as follows. Let F = "$g(Z) < 0$" denote the target failure region and let F_i = "$g(Z) < b_i$" and $b_1 > b_2 > \cdots > b_m = 1$. Therefore, $F_1 \supset F_2 \supset \cdots \supset F_m = F$ are a sequence of m nested failure regions. By the definition of conditional probability,

$$\begin{aligned} P\left(g(Z)<0\right) &= P(F) = P(F_m \mid F_{m-1})P(F_{m-1}) \\ &= \cdots = P(F_1)\prod_{i=2}^{m} P(F_i \mid F_{i-1}) \end{aligned} \tag{10}$$

This equation indicates that instead of directly calculating a small $P(g(Z) < 0)$, one can in principle calculate the probabilities $P(F_1)$, $P(F_i|F_{i-1})$ ($i = 2, \cdots, m$) and then take their product. The potential advantage of this alternative is that the probabilities involved can be chosen to be much greater than $P(g(Z) < 0)$, thus avoiding simulation of rare events.

As indicated in an overview paper by Schuëller, *et al*. (2004), subset simulation is among the most robust stochastic simulation methods for reliability calculations, as quoted:

"Subset simulation has a wide range of applicability. It performs well irrespectively of the geometry and number of the failure domains. It is also applicable to non-Gaussian distributed random variables. It retains the basic advantage of direct Monte Carlo whose performance is unaffected by the dimension of the random parameter vector."

4. RESULTS OF RELIABILITY ANALYSES

Table 2 summarizes the analysis results of the adopted reliability methods. The factor of safety of this problem, defined as S_{allow} divided by the settlement when all uncertain variables are fixed at their mean values, is found to be around 2.1. The reliability index β for FORM is the distance of the obtained design point to the origin. For the other methods, β is simply taken to be $-\Phi^{-1}(P_F)$, where P_F denotes the failure probability estimate, and Φ is the cumulative density function (CDF) of the standard Gaussian distribution. For MCS, 10^6 samples are taken, and for Subsim, 1000 samples are taken in each stage.

The FOSM, FORM, and SORM are analytical methods, so the P_F estimators are deterministic. However, MCS and Subsim are simulation methods: their P_F estimators are random. The cov of their estimates are listed in the table. The cov for MCS is calculated based on the formula cov = $[(1 - P_F)/P_F/n]^{0.5}$, where $n = 10^6$ is the total number of MCS samples. The cov for Subsim is estimated based on the P_F estimates from 100 independent Subsim runs.

Table 2 Primary analysis results

Solution method	FOSM	FORM	SORM	Subsim	MCS
β	2.94	1.43 2.15*	1.55 2.29*	1.53	1.52
P_F estimate	0.0017	0.076 0.016	0.061 0.011	0.063	0.064
% error in P_F estimate	–97.4	19.3 –75.3	–4.7 –82.6	–0.8	–
# of g evaluations	19	152	261	1900	10^6
Estimator cov	n/a	n/a	n/a	10.2%	0.4%

* multiple design point solutions

The percentage error in P_F estimate is the percentage error compared to the MCS solution, which should be very close to the actual failure probability judging from the 0.4% cov. In spite of the high efficiency, FOSM significantly underestimates P_F. The poor performance of FOSM may be due to its simplified assumption on the distribution of the g function and due to the inaccuracy of the linearization for the g function.

For FORM, the adopted gradient projection (GP) algorithm yields two local solutions in the standard Gaussian space: the first one gives a reliability index of 1.43, and the second one gives a reliability index of 2.15. It is clear that the first solution is the global solution because it is closer to the origin. This global solution corresponds to a failure probability of 0.076, which is reasonably close to the MCS solution 0.064. Whether the GP algorithm would converge to the global solution depends on the location of the initial trial point in the standard Gaussian space. Unfortunately, when the initial trial point is randomly generated from the standard Gaussian distribution, there is only about 20% chance of converging to the global solution: 80% of the time the algorithm will converge to the fake solution. In particular, when the initial trial point is taken to be the origin of the standard Gaussian space, the GP algorithm converges to the fake solution.

The SORM uses the solution from FORM, therefore SORM also leads to two solutions. The first solution of $P_F = 0.061$ follows from the global solution of FORM, while the second solution of $P_F = 0.011$ follows from the fake solution of FORM. Note that the first SORM solution is fairly close to the MCS solution 0.064, indicating SORM indeed improves FORM for this case study if the global solution can be found. However, SORM suffers from the same issue of multiple local design points because SORM uses the results from FORM.

The P_F estimate made by Subsim is quite accurate although the required computation is more than those for FOSM, FORM, and SORM.

5. ISSUE OF MULTIPLE DESIGN POINTS

This section discusses in detail the issue of multiple local solutions (design points) of FORM. This issue only affects the analysis results of FORM and SORM; MCS and Subsim are robust against this issue.

5.1 Multiple Solutions

Table 3 lists the coordinates of the two local solutions obtained in the GP algorithm. After verification, it is found that both solutions satisfy the following two necessary conditions for design point:

a. The solution should reside right on the limit-state line $g = 0$.
b. In the standard Gaussian space, the gradient vector of the performance function evaluated at the solution should be parallel to the solution vector itself, *i.e.*, the solution is the point on the $g = 0$ line that is locally closest to the origin.

Moreover, from the numerical values of the coordinates of the two solutions, it is not trivial to identify which solution is global or fake. As a consequence, for the instance that a single run of the GP algorithm converges to the fake solution, there seems to be no viable way at hand to detect the falsity.

Table 3 Coordinates of the two local design points

Component	Coordinates of the design points in the standard Gaussian space	
	Global solution $\beta = 1.43$ $P_F = 0.076$	Fake solution $\beta = 2.15$ $P_F = 0.016$
q	0.82	0.97
H	–0.05	0.39
e^{clay}	0.04	–0.32
e^{sand}	0.14	0.16
G_s^{clay}	–0.10	–0.12
G_s^{sand}	–0.09	–0.10
OCR	–1.09	0.00
C_c	0.35	1.59
α	0.18	0.92

5.2 Non-Differentiability of Performance Function

Figure 2 is employed to demonstrate the geometry around these two local design points. The first plot shows the contour lines of the performance function g in the standard Gaussian space of q and OCR, where all other seven uncertain parameters are fixed at their solution coordinates of the global solution. The second plot is for the fake solution. The marker 'x' indicates the locations of the two local design points. The dark dashed lines represent the non-differentiable boundary for $\sigma' = \sigma'_p$.

The existence of two local design points may have something to do with the non-differentiability of the performance function. Moreover, the failure region in the standard Gaussian space seems to be the union of failure regions defined by the following two performance functions:

$$g_1 = S_{\text{allow}} - \frac{H}{1+e^{\text{clay}}}\left[C_r \log\left(\frac{\sigma'_p}{\sigma'_0}\right) + C_c \log\left(\frac{\sigma'}{\sigma'_p}\right)\right] \quad (11)$$

and

$$g_2 = S_{\text{allow}} - H \cdot C_r \log_{10}\left(\frac{\sigma'}{\sigma'_0}\right) \Big/ 1+e^{\text{clay}} \quad (12)$$

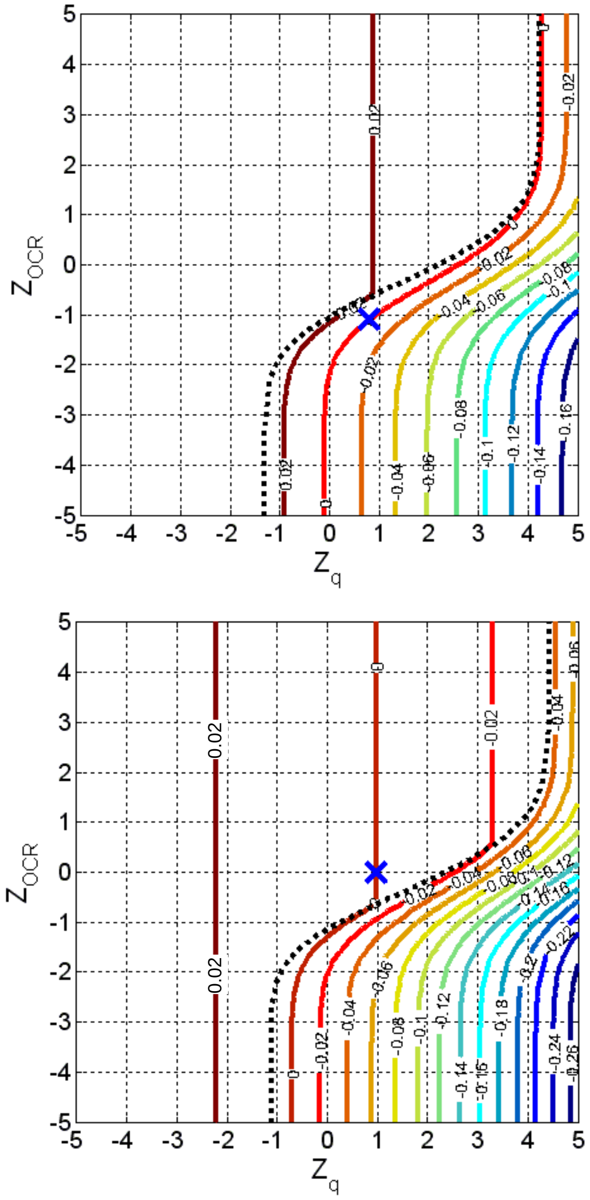

Fig. 2 **The contour lines of the performance function around the two solutions (upper: global solution; lower: fake solution)**

Let us take the region around the second local design point (the lower plot in Fig. 2) as an example. The region in the lower plot in Fig. 2 satisfying $g < 0$ (failure) resides at the right hand side of the $g = 0$ contour line, which is clearly the union of the two failure regions defined by $g_1 < 0$ and $g_2 < 0$ shown in Fig. 3. In other words, the consolidation problem is similar to an in-series system whose failure is defined by the disjunction of the two failure events, *i.e.*, failure occurs either the failure of g_1 or failure of g_2 occurs. It is interesting to see that the original problem in (2) is not in-series, but it behaves like an in-series system due to non-differentiability of the performance function.

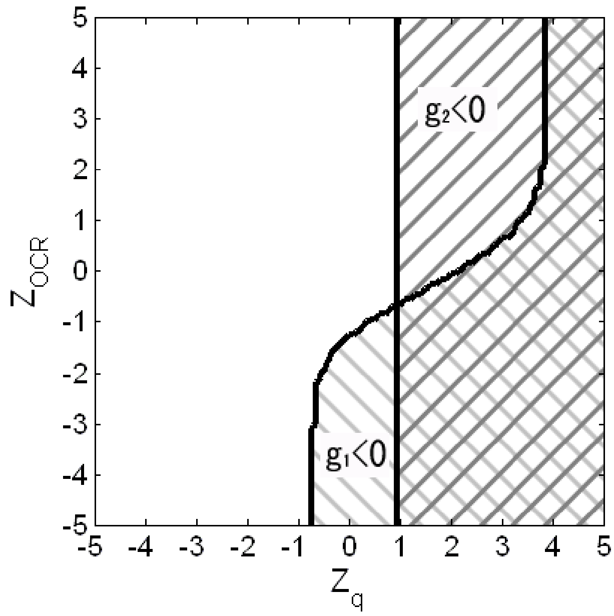

Fig. 3 **The failure regions defined by g_1 and g_2 function around the second local design point**

5.3 Other Design Point Algorithms

Apart from the GP algorithm, another two algorithms of FORM for finding the design point are examined, including the fmincon.m function in matlab® and the simple algorithm in Section 6.2.4, p. 361 in Ang and Tang (1984). Table 4 summarizes the results of convergence. All algorithms may converge to the fake solution, depending on the location of the initial trial point. When the initial trial point is randomly generated from the standard Gaussian distribution, there is always larger chance to converge to the fake solution. When the initial trial point is taken to be the origin, only the matlab algorithm converges to the global solution: it is likely that this only happens by luck. Furthermore, there is a certain chance for non-convergence, *i.e.*, the algorithm never reaches a stationary point. The non-convergence may also be due to the non-differentiability of the performance function.

Table 4 **Convergence for the three algorithms**

Algorithm	Chance of convergence			Convergence if starting from the origin
	Fake solution	Global solution	No converge	
GP	78%	15%	7%	Fake solution
Matlab	70%	28%	2%	Global solution
Ang and Tang	83%	17%	0%	Fake solution

6. SENSITIVITY ANALYSIS

It is instructive to understand how the existence of multiple local design points for FORM/SORM would be affected by the change of the input parameters and distributions. Again, this issue of multiple local design points only affects FORM/SORM; MCS and Subsim are robust against this issue.

6.1 Sensitivity over Mean Value of *q*

Table 5 shows the analysis results by using the GP algorithm for various choices of the mean value of q, denoted by μ_q, while the input values and distributions of other variables are identical to those in Table 1. The chance of non-convergence is based on the case where the initial trial point is randomly generated by the standard Gaussian distribution. It is clear that the existence of multiple solutions only happens for μ_q near 20. Moreover, the chance of non-convergence increases as the failure probability becomes smaller. Compared with the MCS results, the P_F estimated from the global solution of FORM is reasonably accurate, although a bias is noticeable.

6.2 Sensitivity over Range of OCR

Table 6 shows the analysis results by using the GP algorithm for various choices of the range of OCR, while the input values and distributions of other variables are identical to those in Table 1. It is clear that the existence of multiple solutions happens for the range of 0 ~ 10 and 1.5 ~ 2.5. The chance of non-convergence is reasonably small regardless the OCR range. Compared with the MCS results, the P_F estimated from the global solution of FORM is reasonably accurate despite a certain amount of bias.

6.3 Sensitivity over S_{allow}

Table 7 shows the analysis results by using the GP algorithm for various choices of S_{allow}, while the input values and distributions of other variables are identical to those in Table 1. It is clear that multiple solutions occur for several cases. The chance of non-convergence increases as the failure probability gets small. Compared with the MCS results, the P_F estimated from the global solution of FORM is reasonably accurate despite a certain amount of bias.

Table 5 Analysis results for various choices of μ_q

μ_q (kN/m^2)	Chance of convergence (P_F estimate from FORM)			MCS P_F ($n = 10^6$)	P_F 2-order bound	P_F Point estimate
	Fake solution	Global solution	No converge			
5	0% (n/a)	23% (6.2e–9)	77%	0	[6.2e–9 6.2e–9]	6.2e–9
10	0% (n/a)	66% (6.1e–5)	34%	5.6e–5	[6.1e–5 6.1e–5]	6.1e–5
20	78% (0.016)	15% (0.076)	7%	0.064	[0.083 0.087]	0.085
30	0% (n/a)	91% (0.35)	9%	0.37	[0.36 0.40]	0.39
50	0% (n/a)	100% (0.85)	0%	0.88	[0.91 1.00]	0.95

Table 6 Analysis results for various choices of OCR range

OCR range	Chance of convergence (P_F estimate from FORM)			MCS P_F ($n = 10^6$)	P_F 2-order bound	P_F Point estimate
	Fake solution	Global solution	No converge			
0 ~ 10	76% (0.016)	16% (0.15)	8%	0.16	[0.16 0.16]	0.16
0 ~ 1.5	0% (n/a)	100% (0.94)	0%	0.95	[0.92 0.92]	0.92
1.5 ~ 2.5	78% (0.016)	15% (0.076)	7%	0.064	[0.083 0.087]	0.085
2.5 ~ 5	0% (n/a)	98% (0.016)	2%	0.011	[0.016 0.016]	0.016
5 ~ 10	0% (n/a)	93% (0.016)	7%	0.011	[0.016 0.016]	0.016

Table 7 Analysis results for various choices of $S_{\mathbf{allow}}$

S_{allow} (mm)	Chance of convergence (P_F estimate from FORM)			MCS P_F ($n = 10^6$)	P_F 2-order bound	P_F Point estimate
	Fake solution	Global solution	No converge			
1	0% (n/a)	100% (0.98)	0%	0.99	[0.99 1.00]	0.99
3	10% (0.15)	89% (0.24)	1%	0.28	[0.29 0.34]	0.33
5	78% (0.016)	15% (0.076)	7%	0.064	[0.083 0.087]	0.085
10	19% (8.2e–6)	39% (8.4e–3)	42%	4.9e–3	[8.4e–3 8.4e–3]	8.4e–3
15	0% (n/a)	29% (7.2e–4)	71%	3.7e–4	[7.2e–4 7.2e–4]	7.2e–4

7. POSSIBLE REMEDY FOR FORM/SORM

FORM and SORM are considered to be accurate methods among popular reliability methods. They have been widely applied to civil and geotechnical engineering problems. Unfortunately, they may lead to wrong solutions for the consolidation example. It is constructive to propose FORM- or SORM-based methods that always lead to the correct solution. This is the purpose of this section. Readers are alerted that these remedial methods may only be effective for the consolidation problem; they may be not very useful for other geotechnical examples with non-differentiable performance functions that also suffer from the multiple design problems.

As mentioned earlier, the non-in-series system defined in (2) behaves similarly as the in-series system. Therefore, a possible solution to resolve the issue of multiple local solutions of FORM/SORM is to replace the original system by the in-series system. For an in-series system with two failure modes of g_1 and g_2 described in (11) and (12), two methods can be taken: (a) the second-order bound (Ang and Tang, 1984) and (b) point estimate (Mendell and Elston, 1974; Phoon, 2008). These two methods are briefly reviewed herein. Outside the FORM/SORM framework, the issue of multiple local design points can be easily handled by adopting either MCS or Subsim.

7.1 Second-Order Bound

Under the assumption that the two failure modes g_1 and g_2 are positively correlated, it is possible to derive the upper and lower bounds for the failure probability of the consolidation problem by considering the geometrical properties of intersections of the two failure regions, leading to the so-called second-order bound. The second-order bound simply says that the actual failure probability P_F of a system with two in-series positively correlated components should fall into the following upper and lower bounds:

$$P_1 + \max\ [P_2 - P_{21}^{+},\ 0] \le P_F \le \min\ [P_1 + P_2 - P_{21}^{-},\ 1] \quad (13)$$

where $P_i = \Phi(-\| z_i^* \|)$ and z^*_i is the design point for the i-th failure mode: *i.e.*, z^*_1 is the design point if g_1 in (11) is the only performance function (similar for z^*_2), so the process of finding z^*_1 or z^*_2 will not suffer from the issue of multiple solutions; $P^+_{21} = P(B_1) + P(B_2)$ is the upper bound of the probability of $\{B_1$ and $B_2\}$; $P^-_{21} = \max\ [P(B_1), P(B_2)]$ is the lower bound of the probability of $\{B_1$ and $B_2\}$;

$$P(B_1) = \Phi(-\| z_1^* \|) \cdot \Phi\left[\left(\| z_1^* \| \cos(\theta) - \| z_2^* \|\right) / \sin(\theta)\right] \quad (14)$$

and

$$P(B_2) = \Phi(-\| z_2^* \|) \cdot \Phi\left[\left(\| z_2^* \| \cos(\theta) - \| z_1^* \|\right) / \sin(\theta)\right] \quad (15)$$

where θ is the angle between the two design point vectors z^*_1 and z^*_2.

The resulting upper and lower bounds for the cases studied in the previous section are listed in Tables 5 to 7. Slight bias of the bounds can be observed, *i.e.*, MCS results occasionally fall outsides the bounds, but the issue of multiple solutions is resolved because the process of finding z^*_1 or z^*_2 does not involve multiple solutions.

7.2 Point Estimate

The second-order bounds do not offer a single estimate of the failure probability. The following point estimate can mitigate this issue:

$$P_F \approx P_1 + P_2 - P_{21} \quad (16)$$

where P_F is the system failure probability;

$$P_{21} = \Phi\left[\frac{\cos(\theta)\ a_1 - \| z_2^* \|}{\sqrt{1 - \cos^2(\theta)\ a_1 (a_1 - \| z_1^* \|)}}\right] \Phi(-\| z_1^* \|) \quad (17)$$

is an estimate of the probability that g_1 and g_2 failure events happen simultaneously, and

$$a_1 = \frac{1}{\Phi(-\| z_1^* \|)\sqrt{2\pi}}\ \exp(-\| z_2^* \|^2 / 2) \quad (18)$$

The resulting point estimates are also listed in Tables 5 to 7. Again, slight bias of the point estimates is evident, but the issue of multiple solutions is resolved.

8. RECOMMENDED RELIABILITY METHODS

For the consolidation problem, FOSM does not provide a consistent reliability estimate. Moreover, the following two issues exist in FORM/SORM: (a) the possibility of finding a non-global (fake) design point and (b) the possibility of non-convergence. The second-order bound and point estimate methods seem to be able to resolve these issues although the resulting bounds/estimates seem slightly biased. Therefore, the use of the latter two methods is recommended at the condition that the amount of bias is acceptable. This conclusion may be only applicable to the consolidation example and may not be applicable to other geotechnical examples with non-differentiable performance functions.

Monte Carlo simulation and Subset simulation are completely robust against the existence of local design points. Therefore, their use is recommended at the expense of more computation. This conclusion is applicable to all geotechnical examples with non-differentiable performance functions.

9. CONCLUSIONS

The reliability analysis of a one-dimensional consolidation is presented. This “simple” geotechnical problem turns out to be challenging because the performance function is not differentiable. It is found that this non-differentiability may induce a behavior similar to an in-series system although the consolidation problem is clearly not an in-series system.

Several popular reliability methods are examined to investigate their feasibility and consistency over a problem with a non-differentiable performance function. The first-order second-moment (FOSM) method provides inconsistent reliability estimates. For first-order and second-order reliability methods (FORM/SORM), three algorithms of finding design points are examined, and all of them show possibility of converging to a local design point that is different from the global one, hence giving inconsistent reliability estimate. Sometimes they do not converge at all. Therefore, cautions should be used when implementing FORM/SORM to problems with non-differentiable performance functions. The 1-D consolidation example serves as a “warning” example to alert geotechnical researchers and practicing engineers to use cautions in choosing reliability methods.

The second-order bound and point estimate methods can resolve the issue of multiple local design points for FORM/SORM although the resulting estimates are slightly biased. Monte Carlo simulation (MCS) and subset simulation (Subsim) are robust and always provide consistent reliability estimates although they are computational more expensive.

It is recommended that the second-order bound and point estimate methods can be used for the consolidation problem if the amount of bias in reliability estimate is acceptable, but the effectiveness of these methods over other geotechnical examples with non-differentiable performance functions are not unknown. MCS and Subsim are highly recommended because they seem completely robust against the existence of local design points, and this conclusion is general over other geotechnical examples with non-differentiable performance functions.

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